Pergamon

Int. J. Solids Structures Vol. 34, No. 17, pp. 2115-2129, 1997

Printed in Great Britain
0020-7683/97 $17.00 + .00

PII: S0020-7683(96)00136-9

EXACT SOLUTION FOR THE TRANSVERSE VIBRATION OF A BEAM A PART OF WHICH IS A TAPER BEAM AND OTHER PART IS A UNIFORM BEAM

N. M. AUCIELLO
Department of Structural Engineering, University of Basilicata, Via della Tecnica 3, 85100, Potenza, Italy

and

A. ERCOLANO
Department of Industrial Engineering, University of Cassino, Via Zamosch 43, Cassino, Italy

(*Received* 6 *October* 1995; *in revised form* 15 *June* 1996)

Abstract—In this paper, the free-vibration frequencies of tapered beams in the most general possible boundary conditions, are determined by means of direct method. The study is extended to beams, made up of two sections with different cross-sectional variations: the equations of motion can be solved by means of the well-known Bessel function. The natural frequencies were determined using the false position method and symbolic program as roots of the corresponding characteristic equations. Finally an extensive number of structural system previously studied both exactly and approximately by other authors, mentioned in the bibliography, confirms the exactness of the results obtained. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

For many structural elements in actual use it proves to be convenient to study the dynamic behavior, and the quantities that have an influence on the results, by schematizing them: this often leads to the variable cross-section beam model. The results thus obtained supply, by simple analogy, useful indications regarding the influence of the different parameters for complex problem whose solution proves to be difficult to obtain. This simplification, therefore, allows optimization of the structures of mobile arms used in cybernetics, ariels, towers, solar panel frames, tall building etc.

The importance of the problem has been recently highlighted by numerous researchers who have studied the variable cross-section beam by proposing both approximate and exact types of approaches.

In fact, the former have validity applied to complex geometry beams or with discontinuous cross-section, whereas the latter have used for simple schemes. One of the first treatments of the problem in a closed form was by Mabie and Rogers (1968, 1972, 1974), which give the solutions for beams with different bounding constraints and the first five frequencies are determined. Subsequently, an extension to the case of rotational flexibility of the clamped end was formulated by Goel (1976), whereas recently Yang (1980) in a comment to an approximated previous study, analysed the case of a beam with only the height variable. The results by Ward (1913), Conway and Dubil (1965), Sanger (1968), Sato (1980) should also be noted.

For the cone and the wedge beams, an exact solution obtained using the Frobenius method was proposed recently by Naguleswaran (1994) and the results are tabulated for different constraint condition. With reference to more complex structures, and in particular to the case of a shelf with an intermediate elastic support, Craver and Jampala (1993) considered a particular mini-max problem in relation to the stiffness of spring.

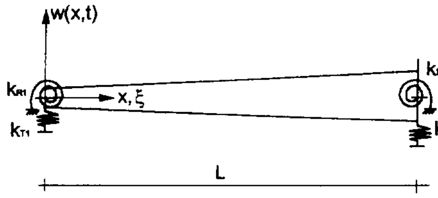


Fig. 1. Tapered beam generally constrained at the ends.

Besides the closed-type solutions obtained, approximated type ones are not rare. Amongst these solutions is To's paper (1982) on the finite element method, which distinguishes itself for its general nature since it is applicable to beams that are generally constrained and for the accuracy of the results obtained. Laura and Gutierrez (1986) and Alvarez *et al.* (1988), instead, to calculate the fundamental frequencies, use the Rayleigh–Ritz procedure and extend it too to the higher frequencies furnished by Grossi and Bhat (1991) and Auciello (1995).

In this paper the natural frequencies are exactly determined for a beam constrained at the ends of different taper ratio of the cross-section. The procedure is analogous to the one described in Mabie and Rogers (1972) in which the solution is expressed in terms of Bessel functions and extended to the case of the beams with cross-sectional discontinuities along the axis. Using a symbolic program (*Mathematica*), Wolfram (1993), also allows the results to be accurately determined.

2. THE BEAM GENERALLY CONSTRAINED AT THE ENDS

Consider the beam in Fig. 1 whose equation of motion, if the Euler–Bernoulli hypotheses are correct, is:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Z(x)}{dx^2}\right]-\rho\omega^2A(x)Z(x)=0 \tag{1}$$

where $Z(x)$ is the transverse displacement, x is abscissa, E is Young's modulus, $I(x)$ is the cross-sectional inertia, $A(x)$ is the cross-sectional area, ρ is the mass density and ω is the natural frequency of vibration.

For a beam in which the height h and the depth b vary linearly according to relationship $\alpha = h(L)/h(0)$, $\zeta = b(L)/b(0)$, the eqn (1) can be written [Mabie and Rogers (1968, 1972)]

$$Z^{IV}+\frac{2}{L}\left[\frac{3(\alpha-1)}{X}+\frac{\zeta-1}{Y}\right]Z^{III}+\frac{6}{L^2}\left[\frac{(\zeta-1)(\alpha-1)}{XY}+\frac{(\alpha-1)^2}{X^2}\right]Z^{II}-\frac{p^4}{L^4X^2}Z=0 \tag{2}$$

where

$$X=(\alpha-1)x/L+1,\quad Y=(\zeta-1)x/L+1,\quad p^4=\frac{\rho A_1}{EI_1}L^4\omega^2 \tag{3a,b,c}$$

and an apix indicates the order of derivation in relation to x. The solutions of (2) are known for $\alpha = \zeta$ and $\zeta = 1$; [Watson (1966)]. In particular, for $\alpha = \zeta$ ($X = Y$), the (2) can be written:

$$X^2\frac{d^4Z}{dX^4}+8X\frac{d^3Z}{dX^3}+12\frac{d^2Z}{dX^2}-\left[\frac{p}{\alpha-1}\right]^4Z=0 \tag{4}$$

and allows solutions of the type

$$Z(X) = X^{-1}[AJ_2(2p_\alpha X^{0.5}) + BY_2(2p_\alpha X^{0.5}) + CI_2(2p_\alpha X^{0.5}) + DK_2(2p_\alpha X^{0.5})] \quad (5)$$

where

$$p_\alpha = \frac{p}{\alpha - 1} \quad (6)$$

and J_2, Y_2, I_2, K_2 are the second-order Bessel functions.

Instead, for the beam in which only the height, $\zeta = 1$ $(Y = 1)$, varies linearly, (2) can be written:

$$X^2\frac{d^4Z}{dX^4} + 6X\frac{d^3Z}{dX^3} + 6\frac{d^2Z}{dX^2} - \left[\frac{p}{\alpha-1}\right]^4 Z = 0 \quad (7)$$

with a general solution

$$Z(X) = X^{-0.5}[AJ_1(2p_\alpha X^{0.5}) + BY_1(2p_\alpha X^{0.5}) + CI_1(2p_\alpha X^{0.5}) + DK_1(2p_\alpha X^{0.5})], \quad (8)$$

where J_1, Y_1, I_1, K_1 are the first-order Bessel function. Equations (5) and (8) can be expressed as a single equation once the parameter n has been defined. In fact, assuming $n = 1$ for only the section height varying and $n = 2$ both the height and the depth varying, one obtains:

$$Z(X) = X^{-n0.5}[AJ_n(2p_\alpha X^{0.5}) + BY_n(2p_\alpha X^{0.5}) + CI_n(2p_\alpha X^{0.5}) + DK_n(2p_\alpha X^{0.5})]. \quad (9)$$

The boundary conditions, expressed in relation to transversal displacement, can be used along side the above-mentioned general solutions.

The boundary conditions can be written:

for $x = 0$

$$k_{T1}Z = (-EIZ^{II})^{I}, \quad k_{R1}Z^{I} = EIZ^{II}, \quad (10, 11)$$

whereas for $x = L$

$$k_{T2}Z = (EIZ^{II})^{I}, \quad k_{R2}Z^{I} = -EIZ^{II}. \quad (12, 13)$$

Substituting eqn (3a, b) into (12–15) the boundary conditions are in $x = 0$

$$-k_{T1}Z = EI_1\left(\frac{\zeta-1}{L} + 3\frac{\alpha-1}{L}\right)Z^{II} + EI_1Z^{III} \quad (14)$$

$$k_{R1}Z^{I} = EI_1Z^{II} \quad (15)$$

and at point in $x = L$

$$k_{T2}Z = \frac{EI_2}{L}\left(\frac{\zeta-1}{\zeta} + 3\frac{\alpha-1}{\alpha}\right)Z^{II} + EI_2Z^{III} \quad (16)$$

$$k_{R2}Z^{I} = -EI_2Z^{II}. \quad (17)$$

The previous equations in term of dimensionless variables (3a, b), bearing in mind that

$$\mathrm{d}x = \frac{L}{\alpha-1}\mathrm{d}X \tag{18}$$

become

$$\frac{k_{T1}L^3}{EI_1}Z+[3\alpha+\zeta-4](\alpha-1)^2\frac{\mathrm{d}^2Z}{\mathrm{d}X^2}+(\alpha-1)^3\frac{\mathrm{d}^3Z}{\mathrm{d}X^3}=0 \tag{19}$$

$$\frac{k_{R1}L}{EI_1}\frac{\mathrm{d}Z}{\mathrm{d}X}-(\alpha-1)\frac{\mathrm{d}^2Z}{\mathrm{d}X^2}=0 \tag{20}$$

$$\frac{k_{T2}L^3}{EI_2}Z-\left[\frac{\zeta-1}{\zeta}+3\frac{\alpha-1}{\alpha}\right](\alpha-1)^2\frac{\mathrm{d}^2Z}{\mathrm{d}X^2}-(\alpha-1)^3\frac{\mathrm{d}^3Z}{\mathrm{d}X^3}=0 \tag{21}$$

$$\frac{k_{R2}L}{EI_2}\frac{\mathrm{d}Z}{\mathrm{d}X}+(\alpha-1)\frac{\mathrm{d}^2Z}{\mathrm{d}X^2}=0. \tag{22}$$

If the general solution is substituted into boundary conditions (19–22) a homogeneous system is obtained in the unknowns A, B, C, D.

Hence, (19) and (20), which correspond to the boundary conditions for $x = 0$, $(X = 1)$, become

$$\begin{aligned}
&A\left[\frac{J_n(a)}{C_{T1}(\alpha-1)^3}+(n+2)p_\alpha^2J_{n+2}(a)-p_\alpha^3J_{n+3}(a)\right]\\
&\quad+B\left[\frac{Y_n(a)}{C_{T1}(\alpha-1)^3}+(n+2)p_\alpha^2Y_{n+2}(a)-p_\alpha^3Y_{n+3}(a)\right]\\
&\quad+C\left[\frac{I_n(a)}{C_{T1}(\alpha-1)^3}+(n+2)p_\alpha^2I_{n+2}(a)+p_\alpha^3I_{n+3}(a)\right]\\
&\quad+D\left[\frac{K_n(a)}{C_{T1}(\alpha-1)^3}+(n+2)p_\alpha^2K_{n+2}(a)-p_\alpha^3K_{n+3}(a)\right]=0,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&A\left[-\frac{p_\alpha}{C_{R1}(\alpha-1)}J_{n+1}(a)-p_\alpha^2J_{n+2}(a)\right]+B\left[-\frac{p_\alpha}{C_{R1}(\alpha-1)}Y_{n+1}(a)-p_\alpha^2Y_{n+2}(a)\right]\\
&\quad+C\left[\frac{p_\alpha}{C_{R1}(\alpha-1)}I_{n+1}(a)-p_\alpha^2I_{n+2}(a)\right]+D\left[-\frac{p_\alpha}{C_{R1}(\alpha-1)}K_{n+1}(a)-p_\alpha^2K_{n+2}(a)\right]=0
\end{aligned} \tag{24}$$

where it is assumed that:

$$a = 2p_\alpha,\quad C_{T1}=\frac{EI_1}{k_{T1}L^3},\quad C_{R1}=\frac{EI_1}{k_{R1}L} \tag{25}$$

whereas (21) and (22) in $x = L$, $(X = \alpha)$ are

$$\begin{aligned}
&A\left[\frac{\alpha^2}{C_{T2}(\alpha-1)^3}J_n(\alpha\alpha)-(n+2)p_\alpha^2J_{n+2}(\alpha\alpha)+p_\alpha^3\alpha^{0.5}J_{n+3}(\alpha\alpha)\right]\\
&\quad+B\left[\frac{\alpha^2}{C_{T2}(\alpha-1)^3}Y_n(\alpha\alpha)-(n+2)p_\alpha^2Y_{n+2}(\alpha\alpha)+p_\alpha^3\alpha^{0.5}Y_{n+3}(\alpha\alpha)\right]
\end{aligned}$$

$$+C\left[\frac{\alpha^2}{C_{T2}(\alpha-1)^3}I_n(\alpha\alpha)-(n+2)p_\alpha^2 I_{n+2}(\alpha\alpha)-p_\alpha^3\alpha^{0.5}I_{n+3}(\alpha\alpha)\right]$$

$$+D\left[\frac{\alpha^2}{C_{T2}(\alpha-1)^3}K_n(\alpha\alpha)-(n+2)p_\alpha^2 K_{n+2}(\alpha\alpha)+p_\alpha^3\alpha^{0.5}K_{n+3}(\alpha\alpha)\right]=0 \tag{26}$$

$$A\left[-\frac{\alpha^{0.5}}{C_{R2}(\alpha-1)}J_{n+1}(\alpha\alpha)+p_\alpha J_{n+2}(\alpha\alpha)\right]+B\left[-\frac{\alpha^{0.5}}{C_{R2}(\alpha-1)}Y_{n+1}(\alpha\alpha)+p_\alpha Y_{n+2}(\alpha\alpha)\right]$$

$$+C\left[\frac{\alpha^{0.5}}{C_{R2}(\alpha-1)}I_{n+1}(\alpha\alpha)+p_\alpha I_{n+2}(\alpha\alpha)\right]+D\left[-\frac{\alpha^{0.5}}{C_{R2}(\alpha-1)}K_{n+1}(p_\alpha\alpha^{0.5})+p_\alpha K_{n+2}(\alpha\alpha)\right]=0 \tag{27}$$

with

$$C_{T2}=\frac{EI_2}{k_{T2}L^3},\quad C_{R2}=\frac{EI_2}{k_{R2}L},\quad \alpha\alpha=2p_\alpha\alpha^{0.5}. \tag{28}$$

For a non-trivial solution of the system constituted by (23, 24, 26, 27) the determinant of the coefficient matrix is set equal to zero, yielding the frequency equation

$$\det[\alpha_{ij}]=0. \tag{29}$$

The roots of (29) represent the free frequencies of the problem. These were calculated using the false position method. The frequencies are expressed in dimensionless terms and are obtained operating on (29) making use of a symbolic calculation program to manage the Bessel functions.

3. CROSS-SECTION DISCONTINUITIES ALONG THE BEAM

Using the results of the previous paragraphs, it is possible to analyse the problem of a beam made up of two parts of different cross-section. In particular, given the current use of this schema, the variable cross-section beam shown in Fig. 2 is considered: this was obtained by assembling, in a continuous manner, a constant cross-section beam with a linearly variable cross-section beam [Laura and Gutierrez (1986)]. In this case the equations of motion, assuming

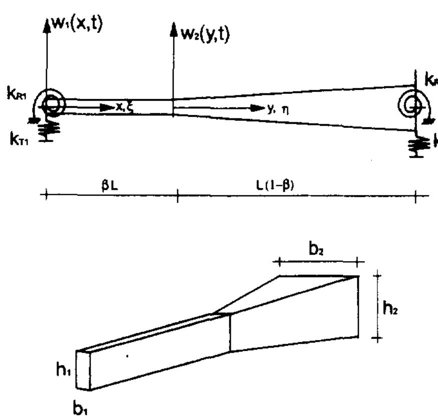


Fig. 2. Beam with cross-section discontinuities.

$$w_1(x,t) = R(x)\,e^{i\omega t} \quad 0 \leqslant x \leqslant \beta L$$

$$w_2(y,t) = S(y)\,e^{i\omega t} \quad 0 \leqslant y \leqslant (1-\beta)L \tag{30}$$

can be written:

$$EI_1 \frac{d^4 R}{dx^4} - \rho A_1 \omega^2 R = 0, \quad 0 \leqslant x \leqslant \beta L \tag{31}$$

$$\frac{d^2}{dy^2}\left[EI_y \frac{d^2 S}{dy^2}\right] - \rho A_y \omega^2 S = 0, \quad 0 \leqslant y \leqslant (1-\beta)L. \tag{32}$$

It is possible to render the problem dimensionless by introducing the parameters

$$\xi = \frac{x}{L}, \quad \eta = \left[1 + \frac{\alpha - 1}{L(1-\beta)} y\right] \tag{33,34}$$

and therefore, for a section whose cross-section varies linearly, the area and the inertia can be generally written to the laws:

$$A_y = A_1 \eta^n; \quad I_y = I_1 \eta^{n+2} \tag{35,36}$$

corresponding, as usual, to $n = 1$ if

$$\alpha = \frac{h_2}{h_1}, \quad \frac{b_2}{b_1} = 1$$

and $n = 2$ for

$$\alpha = \frac{h_2}{h_1} = \frac{b_2}{b_1}.$$

Consequently the equations of motion are

$$\frac{d^4 R}{d\xi^4} - p^4 R = 0, \quad 0 \leqslant \xi \leqslant 1 \tag{37}$$

$$\eta^2 \frac{d^4 S}{d\eta^4} + 8\eta \frac{d^3 S}{d\eta^3} + 12 \frac{d^2 S}{d\eta^2} - q_\alpha^4 S = 0, \quad 1 \leqslant \eta \leqslant \alpha, \quad \rightarrow n = 2 \tag{38a}$$

$$\eta^2 \frac{d^4 S}{d\eta^4} + 6\eta \frac{d^3 S}{d\eta^3} + 6 \frac{d^2 S}{d\eta^2} - q_\alpha^4 S = 0, \quad 1 \leqslant \eta \leqslant \alpha, \quad \rightarrow \eta = 1 \tag{38b}$$

where

$$q_\alpha = p \frac{1-\beta}{\alpha - 1}.$$

The solution to eqns (37) and (38) can be expressed:

$$R(\xi) = A\cosh p\xi + B\sinh p\xi + C\cos p\xi + D\sin p\xi \tag{39}$$

$$S(\eta) = \eta^{-n0.5}[EJ_n(2q_\alpha\eta^{0.5}) + FY_n(2q_\alpha\eta^{0.5}) + GI_n(2q_\alpha\eta^{0.5}) + HK_n(2q_\alpha\eta^{0.5})], \tag{40}$$

where $A, B, \ldots, H$ are the eight integration constants.

It is necessary to accompany the general solutions with the boundary ones, which expressed in terms of dimensionless coordinates, can be represented:

in $x = 0$

$$R + C_{T1}\frac{\mathrm{d}^3R}{\mathrm{d}\xi^3} = 0, \quad \frac{\mathrm{d}R}{\mathrm{d}\xi} - C_{R1}\frac{\mathrm{d}^2R}{\mathrm{d}\xi^2} = 0 \tag{41,42}$$

in $x = \beta L$ for the continuity condition

$$R = S, \quad \frac{\mathrm{d}R}{\mathrm{d}\xi} - \frac{\alpha-1}{1-\beta}\frac{\mathrm{d}S}{\mathrm{d}\eta} = 0, \quad \frac{\mathrm{d}^2R}{\mathrm{d}\xi^2} - \left[\frac{\alpha-1}{1-\beta}\right]^2\frac{\mathrm{d}^2S}{\mathrm{d}\eta^2} = 0$$

$$\frac{\mathrm{d}^3R}{\mathrm{d}\xi^3} - (n+2)\left[\frac{\alpha-1}{1-\beta}\right]^3\frac{\mathrm{d}^2S}{\mathrm{d}\eta^2} - \left[\frac{\alpha-1}{1-\beta}\right]^3\frac{\mathrm{d}^3S}{\mathrm{d}\eta^3} = 0 \tag{43}$$

and in correspondence to the right hand extremity, $y = L(1-\beta)$

$$\frac{\mathrm{d}S}{\mathrm{d}\eta} + \frac{\alpha-1}{1-\beta}C_{R2}\frac{\mathrm{d}^2S}{\mathrm{d}\eta^2} = 0, \quad S - \frac{n+2}{\alpha}C_{T2}\left[\frac{\alpha-1}{1-\beta}\right]^3\frac{\mathrm{d}^2S}{\mathrm{d}\eta^2} - C_{T2}\left[\frac{\alpha-1}{1-\beta}\right]^3\frac{\mathrm{d}^3S}{\mathrm{d}\eta^3} = 0. \tag{44}$$

Substituting eqns (39, 40) with their successive derivatives, into the previous relationships, the corresponding characteristic equations whose roots represent the free frequencies can be obtained

$$\det\begin{bmatrix} 1 & C_{T1}p^3 & 1 & -C_{T1}p^3 & 0 & 0 & 0 & 0 \\ -p^2C_{R1} & p & C_{R1}p^2 & p & 0 & 0 & 0 & 0 \\ \cosh p\beta & \sinh p\beta & \cos p\beta & \sin p\beta & -J_n(a) & -Y_n(a) & -I_n(a) & -K_n(a) \\ \sinh p\beta & \cosh p\beta & -\sin p\beta & \cos p\beta & J_{n+1}(a) & Y_{n+1}(a) & -I_{n+1}(a) & K_{n+1}(a) \\ \cosh p\beta & \sinh p\beta & -\cos p\beta & -\sin p\beta & -J_{n+2}(a) & -Y_{n+2}(a) & -I_{n+2}(a) & -K_{n+2}(a) \\ p\sinh p\beta & p\cosh p\beta & p\sin p\beta & -p\cos p\beta & \alpha_{65} & \alpha_{66} & \alpha_{67} & \alpha_{68} \\ 0 & 0 & 0 & 0 & \alpha_{75} & \alpha_{76} & \alpha_{77} & \alpha_{78} \\ 0 & 0 & 0 & 0 & \alpha_{85} & \alpha_{86} & \alpha_{87} & \alpha_{88} \end{bmatrix} = 0 \tag{45}$$

with

$$\alpha_{65} = pJ_{n+3}(a)-(n+2)Q_1J_{n+2}(a), \quad \alpha_{66} = pY_{n+3}(a)-(n+2)Q_1Y_{n+2}(a)$$

$$\alpha_{67} = -pI_{n+3}(a)-(n+2)Q_1I_{n+2}(a), \quad \alpha_{68} = pK_{n+3}(a)-(n+2)Q_1K_{n+2}(a)$$

$$\alpha_{75} = pC_{R2}J_{n+2}(\alpha\alpha)-\alpha^{0.5}J_{n+1}(\alpha\alpha), \quad \alpha_{76} = pC_{R2}Y_{n+2}(\alpha\alpha)-\alpha^{0.5}Y_{n+1}(\alpha\alpha)$$

$$\alpha_{77} = pC_{R2}I_{n+2}(\alpha\alpha)+\alpha^{0.5}I_{n+1}(\alpha\alpha), \quad \alpha_{78} = pC_{R2}K_{n+2}(\alpha\alpha)-\alpha^{0.5}K_{n+1}(\alpha\alpha)$$

$$\alpha_{85} = \alpha^2 J_n(\alpha\alpha)-(n+2)C_{T2}\frac{\alpha-1}{1-\beta}p^2J_{n+2}(\alpha\alpha)+C_{T2}p^3\alpha^{0.5}J_{n+3}(\alpha\alpha)$$

$$\alpha_{86} = \alpha^2 Y_n(\alpha\alpha)-(n+2)C_{T2}\frac{\alpha-1}{1-\beta}p^2Y_{n+2}(\alpha\alpha)+C_{T2}p^3\alpha^{0.5}Y_{n+3}(\alpha\alpha)$$

$$\alpha_{87} = \alpha^2 I_n(\alpha\alpha)-(n+2)C_{T2}\frac{\alpha-1}{1-\beta}p^2I_{n+2}(\alpha\alpha)-C_{T2}p^3\alpha^{0.5}I_{n+3}(\alpha\alpha)$$

$$\alpha_{88} = \alpha^2 K_n(\alpha\alpha)-(n+2)C_{T2}\frac{\alpha-1}{1-\beta}p^2K_{n+2}(\alpha\alpha)+C_{T2}p^3\alpha^{0.5}K_{n+3}(\alpha\alpha),$$

where

$$a = 2q_\alpha, \quad Q_1 = \frac{\alpha-1}{1-\beta}, \quad \alpha\alpha = 2q_\alpha\alpha^{0.5}.$$

4. APPLICATIONS

Equations (29, 45) allow the dimensionless coefficients p_i of the frequencies for different constraint conditions to be calculated for variations in the parameters C_{R1}, C_{T1}, C_{R2}, C_{T2}. For transfer limit values the contraint conditions shown as follows can occur:

$C_{R1} \to \infty, C_{T1} \to \infty; C_{R2} \to \infty, C_{T2} \to \infty$ free-free

$C_{R1} = 0, C_{T1} \to \infty; C_{R2} \to \infty, C_{T2} \to \infty$ sliding-free

$C_{R1} = 0, C_{T1} = 0; C_{R2} \to \infty, C_{T2} \to \infty$ clamped-free

$C_{R1} \to \infty, C_{T1} = 0; C_{R2} \to \infty, C_{T2} \to \infty$ pinned-free

$C_{R1} \to \infty, C_{T1} = 0; C_{R2} \to \infty, C_{T2} = 0$ pinned-pinned

$C_{R1} = 0, C_{T1} = 0; C_{R2} = 0, C_{T2} = 0$ clamped-clamped

$C_{R1} = 0, C_{T1} = 0; C_{R2} \to \infty, C_{T2} = 0$ clamped-pinned

$C_{R1} = 0, C_{T1} = 0; C_{R2} = 0, C_{T2} \to \infty$ clamped-sliding

$C_{R1} = 0, C_{T1} \to \infty; C_{R2} = 0, C_{T2} \to \infty$ sliding-sliding

$C_{R1} = 0, C_{T1} \to \infty; C_{R2} \to \infty, C_{T2} = 0$ sliding-pinned.

The coefficients p_i for limited number of constraint conditions are hereinafter determined briefly. As a first numerical example, the structural schema Fig. 2 should be considered and the first four free frequencies determined for $\alpha = \zeta = 1.5$ as β and C_{T1}, C_{T2} vary. The results are shown in Fig. 3, in three-dimensional graphs to better highlight the parameters under consideration.

Analogously, Fig. 4 shows the results diagrammatically as the stiffness rotational of the constraints varies. It must be highlighted how in the latter case, for $\beta = 0$, the approximate values p_i are obtained by Grossi and Bhat (1991) using the Rayleigh–Ritz method. It can noted, given the nature of the procedure [Clough and Penzien (1982)], the approximate values are greater than the exact ones and from a practical point of view much nearer exact

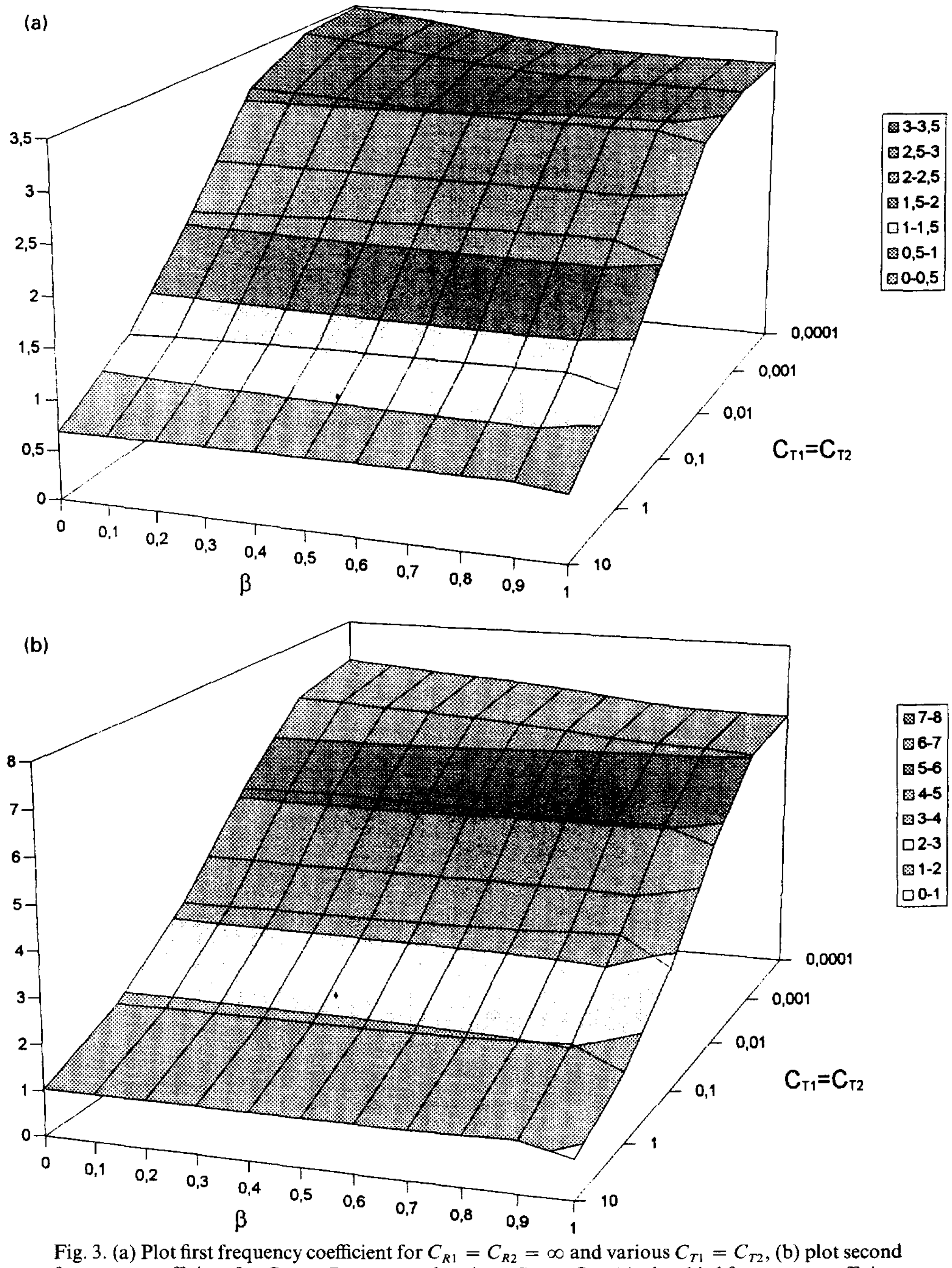
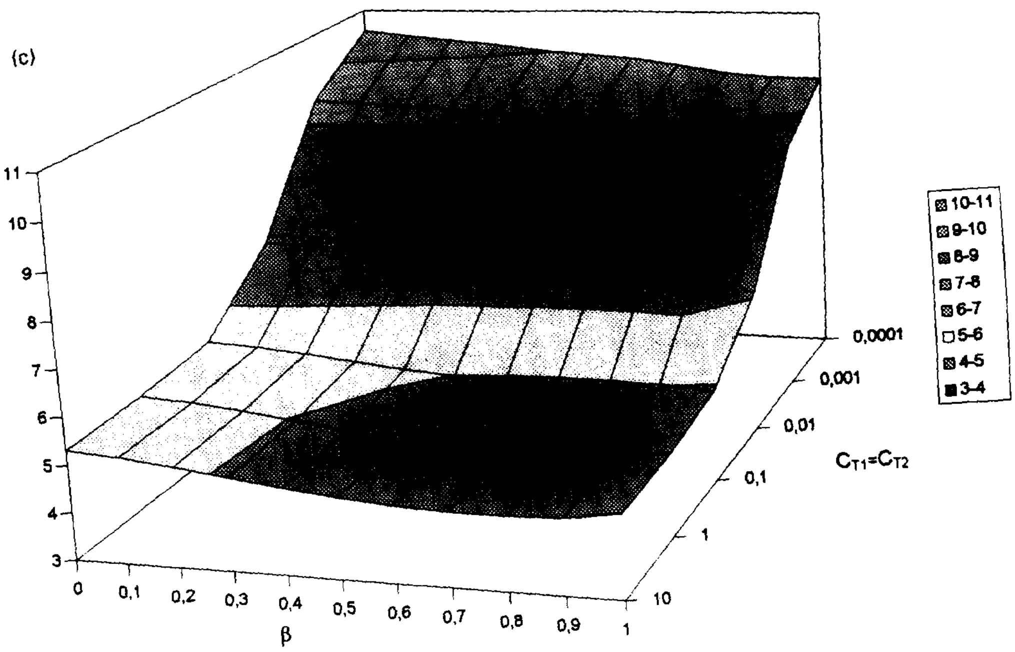


Fig. 3. (a) Plot first frequency coefficient for $C_{R1} = C_{R2} = \infty$ and various $C_{T1} = C_{T2}$, (b) plot second frequency coefficient for $C_{R1} = C_{R2} = \infty$ and various $C_{T1} = C_{T2}$, (c) plot third frequency coefficient for $C_{R1} = C_{R2} = \infty$ and various $C_{T1} = C_{T2}$, (d) plot fourth frequency coefficient for $C_{R1} = C_{R2} = \infty$ and various $C_{T1} = C_{T2}$. (*Continued overleaf.*)

ones. This is due to the accuracy of the approximating functions proposed by Grossi and Bhat (1991). If, for the variable cross-section, variation only in height $n = 1$ is proposed, one obtains the p_i shown in Tables 1–3 corresponding to the different constraint conditions and for different values of the parameter β. For $\beta = 0$, the results, marked with the asterisk, concur perfectly with the ones obtained by Naguleswaran (1994) and unlike the latter, are not numerically influenced by the tapering ratio. For the beam with cross-sectional discontinuities, apart from the case of the cantilever beam in Laura and Gutierrez (1986), the authors do not know of any studies dealing with this aspect, therefore a comparison was not possible.



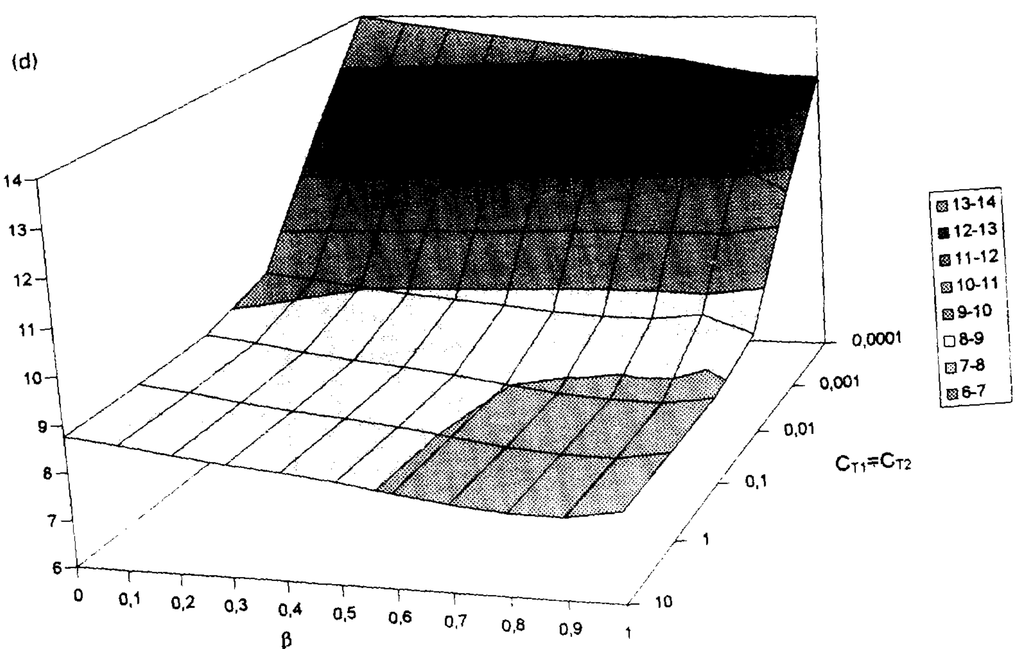


Fig. 3—*Continued.*

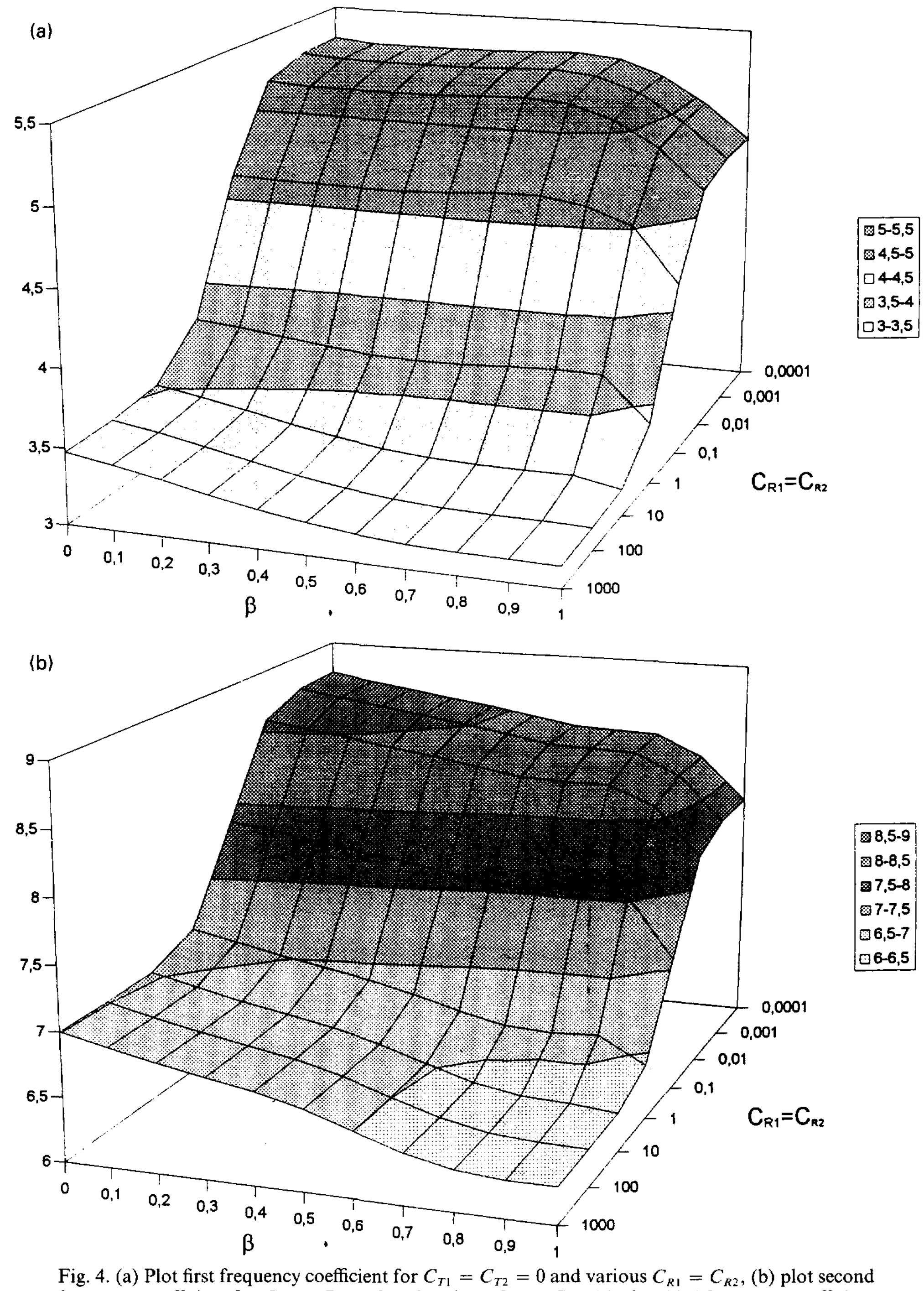
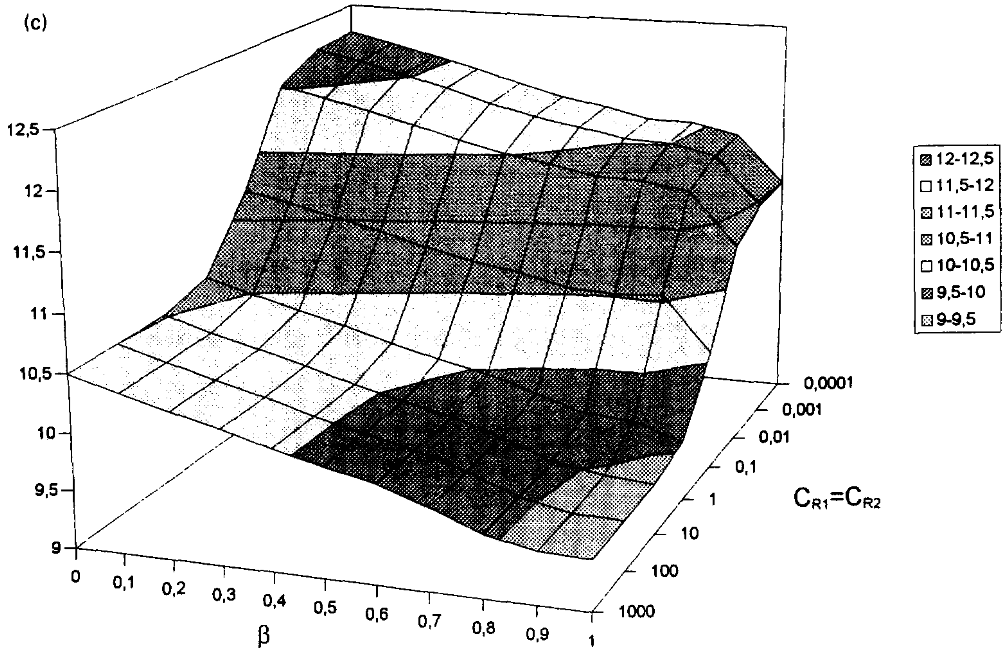


Fig. 4. (a) Plot first frequency coefficient for $C_{T1} = C_{T2} = 0$ and various $C_{R1} = C_{R2}$, (b) plot second frequency coefficient for $C_{T1} = C_{T2} = 0$ and various $C_{R1} = C_{R2}$, (c) plot third frequency coefficient for $C_{T1} = C_{T2} = 0$ and various $C_{R1} = C_{R2}$, (d) plot fourth frequency coefficient for $C_{T1} = C_{T2} = 0$ and various $C_{R1} = C_{R2}$. (*Continued overleaf.*)



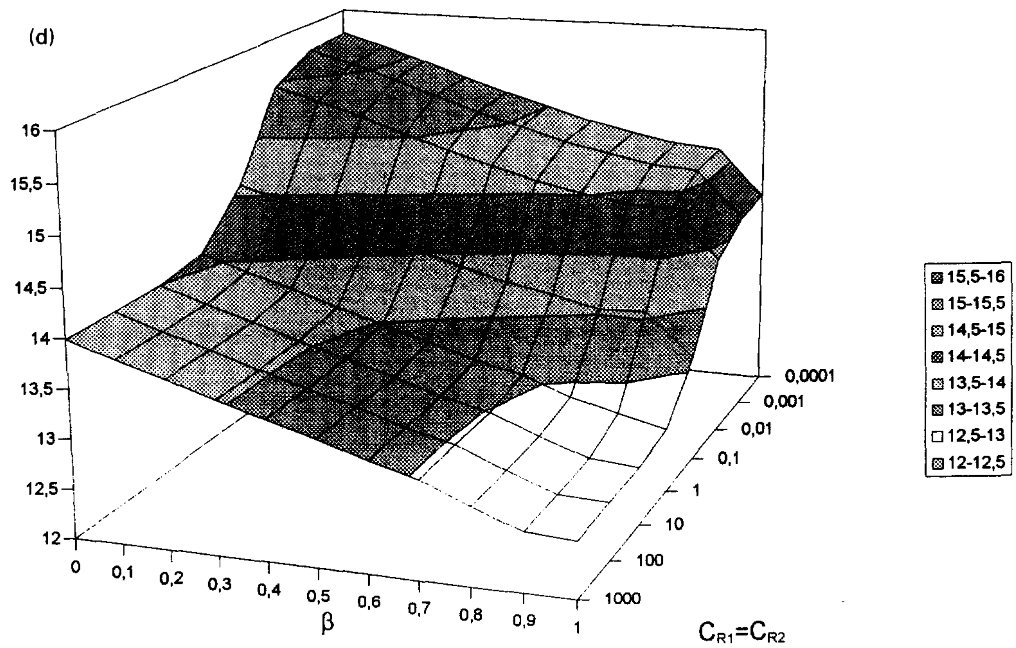


Fig. 4—*Continued.*

Table 1. Frequency coefficients p_i, $i = 1$–5; clamped-clamped for combined beam

α	$\beta = 0$ 1	2	3	4	5	$\beta = 0.2$ 1	2	3	4	5
1	4.7300	7.8532	10.9956	14.1372	17.2788	—	—	—	—	—
1.25	5.0098	8.3172	11.6449	14.9718	18.2988	4.9828	8.2468	11.5303	14.8165	18.1036
1.43	5.1933*	8.6210	12.0699	15.5179	—	—	—	—	—	—
1.5	5.2636	8.7373	12.2325	15.7268	19.2213	5.2104	8.5986	12.0071	15.4214	18.8356
1.54	5.3007*	8.7988	12.3184	15.8373	—	—	—	—	—	—
1.66	5.4215*	8.9985*	12.5975	16.1958	—	—	—	—	—	—
1.75	5.4976	9.1242	12.7732	16.4215	20.0700	5.4186	8.9189	12.4404	15.9700	19.4973
2	5.7159*	9.4848*	13.2769*	17.0684	20.7145	5.6112	9.2146	12.8398	16.4741	20.1029
2.25	5.9213	9.8238	13.7502	17.6761	21.6024	5.7910	9.4904	13.2118	16.9418	20.6627
2.5	6.1159	10.1447	14.1981	18.2512	22.3047	5.9601	9.7498	13.5609	17.3789	21.1841
2.75	6.3012	10.4501	14.6243	18.7983	22.9727	6.1199	9.9954	13.8907	17.7899	21.6727
3	6.4785	10.7421	15.0317	19.3211	23.6112	6.2719	10.2293	14.2038	18.1780	22.1329
4	7.1242*	11.8048*	16.5134*	21.2222	25.9321	6.8185	11.0756	15.3250	19.5488	23.7544
5	7.6947*	12.7427*	17.8202*	22.8984	27.9780	7.2960	11.8213	16.2894	20.7025	25.1251
10	9.9421*	16.4342*	22.9582*	29.4844	36.0136	9.2302	14.7957	19.7536	24.8107	30.1851
			$\beta = 0.4$					$\beta = 0.6$		
1.25	4.9557	8.1761	11.4107	14.6498	17.9006	4.9355	8.1007	11.3025	14.4934	17.7072
1.5	5.1583	8.4647	11.7714	15.0992	18.4409	5.1286	8.3130	11.5711	14.8030	18.0716
1.75	5.3450	8.7264	12.0917	15.5022	18.9192	5.3117	8.5015	11.8086	15.0804	18.3887
2	5.5203	8.9661	12.3812	15.8689	19.3490	5.4852	8.6737	12.0202	15.3335	18.6705
2.25	5.6874	9.1869	12.6465	16.2057	19.7399	5.6491	8.8350	12.2099	15.5672	18.9256
2.5	5.8481	9.3913	12.8926	16.5173	20.0994	5.8032	8.9891	12.3813	15.7842	19.1603
2.75	6.0040	9.5812	12.1233	16.8069	20.4332	5.9473	9.1383	12.5373	15.9862	19.3793
3	6.1559	9.7581	13.3414	17.0773	20.7456	6.0814	9.2844	12.6805	16.1745	19.5859
4	6.7646	10.3592	14.1231	18.0024	21.8410	6.5221	9.8517	13.1664	16.8069	20.3242
5	7.2772	10.8334	14.8061	18.7435	22.7672	6.8326	10.3908	13.5838	17.2834	20.9536
10	9.4280	12.4761	17.2360	21.3717	25.8612	7.4616	12.0636	15.7599	22.7572	26.8834

Table 2. Frequency coefficients p_i, $i = 1$–5; sliding-clamped for combined beam

α	$\beta = 0$ 1	2	3	4	5	$\beta = 0.2$ 1	2	3	4	5
1	2.3650	5.4978	8.6394	11.7810	14.9226	—	—	—	—	—
1.25	2.5804	5.8569	9.1714	12.4925	15.8161	2.5715	5.8095	9.0779	12.3533	15.6350
1.43	2.7249*	6.0943*	9.5211	12.9592	16.4016	—	—	—	—	—
1.5	2.7809	6.1856	9.6552	13.1379	16.6256	2.7671	6.0915	9.4685	12.8610	16.2675
1.54	2.8106*	6.2340*	9.7261*	13.2325	16.7441	—	—	—	—	—
1.66	2.9079*	6.3916*	9.9568*	13.5396	17.1290	—	—	—	—	—
1.75	2.9696	6.4912	10.1023	13.7331	17.3714	2.9539	6.3504	9.8223	13.3202	16.8402
2	3.1487*	6.7783*	10.5204*	14.2886	18.0667	3.1333	6.5907	10.1470	13.7413	17.3663
2.25	3.3198	7.0503	10.9148	14.8115	18.7208	3.3063	6.8152	10.4477	14.1317	17.8545
2.5	3.4838	7.3095	11.2891	15.3072	19.3403	3.4736	7.0264	10.7284	14.4969	18.3114
2.75	3.6418	7.5579	11.6465	15.7796	19.9302	3.6358	7.2258	10.9921	14.8409	18.7415
3	3.7943	7.7967	11.9891	16.2319	20.4947	3.7934	7.4148	11.2412	15.1667	19.1487
4	4.3602*	8.6769*	13.2431*	17.8825	22.5514	4.3835	8.0857	12.1244	16.3315	20.5955
5	4.8713*	9.4673*	14.3595*	19.3456	24.3708	4.9210	8.6510	12.8804	17.3378	21.8246
10	6.9480*	12.6852*	18.8366*	25.1643	31.5730	7.0710	10.5822	15.8160	21.0999	26.1618
			$\beta = 0.4$					$\beta = 0.6$		
1.25	2.5636	5.7558	8.9777	12.2234	15.4602	2.5388	5.7071	8.9030	12.0994	15.2952
1.5	2.7502	5.9807	9.2792	12.6090	15.9291	2.6901	5.9002	9.1282	12.3798	15.6158
1.75	2.9255	6.1811	9.5536	12.9513	16.3482	2.8210	6.0830	9.3254	12.6301	15.9008
2	3.0903	6.3633	9.8064	13.2595	16.7293	2.9341	6.2580	9.5024	12.8551	16.1599
2.25	3.2450	6.5315	10.0414	13.5403	17.0800	3.0317	6.4263	9.6647	13.0582	16.3992
2.5	3.3902	6.6889	10.2611	13.7988	17.4056	3.1161	6.5880	9.8165	13.2422	16.6223
2.75	3.5264	6.8380	10.4673	14.0391	17.7097	3.1893	6.7428	9.9611	13.4096	16.8315
3	3.6541	6.9807	10.6613	14.2642	17.7949	3.2530	6.8906	10.1008	13.5626	17.0281
4	4.0888	7.75117	11.3341	15.0579	18.9820	3.4380	7.4042	10.6381	14.0683	17.7029
5	4.4212	8.0113	11.8735	15.7446	19.7776	3.5528	7.7937	11.1657	14.4729	18.2237
10	5.2300	10.2755	13.5192	18.3764	22.3988	3.7721	8.6284	13.1209	16.4724	19.6841

Table 3. Frequency coefficients p_i, $i = 1$–5; pinned-pinned for combined beam

α	$\beta = 0$ 1	2	3	4	5	$\beta = 0.2$ 1	2	3	4	5
1	3.1416	6.2832	9.4248	12.5664	15.7080	—	—	—	—	—
1.25	3.3253	6.6563	9.9831	13.3098	16.6366	3.2763	6.5687	9.8609	13.1538	16.4428
1.43	3.4438*	6.9026*	10.3504	13.7978	17.2454	—	—	—	—	—
1.5	3.4888	6.9972	10.4912	13.9848	17.4785	3.3886	6.8261	10.2524	13.6780	17.0952
1.54	3.5125*	7.0473*	10.5658	14.0837	17.6018	—	—	—	—	—
1.66	3.5892*	7.2104*	10.8081*	14.4050	18.0023	—	—	—	—	—
1.75	3.6373	7.3133	10.9609	14.6076	18.2546	3.4843	7.0624	10.6101	14.1540	17.6846
2	3.7740*	7.6095*	11.3999*	15.1889	18.9783	3.5668	7.2825	10.9411	14.5915	18.2243
2.25	3.9013	7.8891	11.8137	15.7361	19.6592	3.6386	7.4894	11.2504	14.9974	18.7231
2.5	4.0208	8.1548	12.2062	16.2548	20.3042	3.7017	7.6857	11.5415	15.3765	19.1880
2.75	4.1336	8.4085	12.5806	16.7490	20.9184	3.7573	7.8730	11.8169	15.7326	19.6240
3	4.2408	8.6517	12.9391	17.2220	21.5060	3.8065	8.0527	12.0787	16.0688	20.0353
4	4.6257*	9.5411*	14.2481*	18.9464	23.6462	3.9550	8.7144	13.0170	17.2545	21.4900
5	4.9597*	10.3307*	15.4081*	20.4721	25.5378	4.0492	9.3117	13.8225	18.2528	22.7294
10	6.2366*	13.4623*	20.0028*	26.5031	33.0026	4.1675	11.7968	16.6742	21.8793	27.3378
	$\beta = 0.4$					$\beta = 0.6$				
1.25	3.2169	6.5001	9.7487	12.9980	16.2534	3.1687	6.4143	9.6396	12.8448	19.2768
1.5	3.2727	6.6954	10.0311	13.3783	16.7277	3.1863	6.5173	9.8292	13.0857	19.6466
1.75	3.3150	6.8736	10.2826	13.7204	17.1478	3.1978	6.5997	9.9982	13.3009	19.9738
2	3.3472	7.0376	10.5106	14.0324	17.5256	3.2051	6.6665	10.1496	13.4976	20.2671
2.25	3.3719	7.1893	10.7202	14.3195	17.8695	3.2094	6.7212	10.2857	13.6802	20.5319
2.5	3.3907	7.3302	10.9151	14.5855	18.1860	3.2114	6.7664	10.4083	13.8514	20.7722
2.75	3.4049	7.4614	11.0984	14.8330	18.4801	3.2120	6.8040	10.5188	14.0132	20.9912
3	3.4154	7.5839	11.2721	15.0642	18.7555	3.2112	6.8354	10.6185	14.1665	21.1917
4	3.4322	7.9991	11.9004	15.8560	19.7228	3.1996	6.9187	10.9305	14.7066	21.8531
5	3.4255	8.3191	12.4610	16.4874	20.5450	3.1810	6.9611	11.1429	15.1411	22.3725
10	3.2939	9.1423	14.6626	18.6299	23.3599	3.0649	6.9811	11.5841	16.2371	24.4792

5. CONCLUSIONS

The treatment of the direct method for calculating the free frequencies of tapering beams was extended to the case of beams made up of a constant cross-section and one of linearly variable cross-section. Using a symbolic calculation program noticeably reduced the approximation errors that are normally encountered in this problem, whether one acts in an exact manner or uses an approximate Rayleigh–Ritz-type procedure. This treatment has, as its primary objective, to determine the influence of the different parameters with the aim of optimising the structure from the point of view of its dynamic behavior.

REFERENCES

Alvarez, S. I., Ficcadenti De Iglesias, G. M. and Laura, P. A. A. (1988). Vibrations of an elastically restrained, non-uniform beam with translational springs, and tip mass. *Journal of Sound Vibration* **120,** 465–471.

Auciello, N. M. (1995). A comment on: A note on vibrating tapered beams. *Journal of Sound Vibration* **187,** 724–726.

Clough, R. W. and Penzien, J. (1982). *Dynamics of Structures*. McGraw-Hill, New York.

Conway, H. D. and Dubil, J. F. (1965). Vibration frequencies of truncated cones and wedge beams. *Journal of Applied Mechanics (ASME)* **31,** 329–331.

Craver, W. L., Jr. and Jampala, P. (1993). Transverse vibrations of a linearly tapered cantilever beam with constraining springs. *Journal of Sound Vibration* **166,** 521–529.

Goel, R. P. (1976). Transverse vibrations of tapered beams. *Journal of Sound Vibration* **47,** 1–7.

Grossi, R. O. and Bhat, R. B. (1991). A note on vibrating tapered beams. *Journal of Sound Vibration* **147,** 174–178.

Klein, L. (1974). Transverse vibrations of non-uniform beams. *Journal of Sound Vibration* **37,** 491–505.

Laura, P. A. A. and Gutierrez, R. H. (1986). Vibrations of an elastically restrained cantilever beam of varying cross section with tip mass of finite length. *Journal of Sound Vibration* **108,** 123–131.

Mabie, H. H. and Rogers, C. B. (1974). Transverse vibrations of double-tapered cantilever beams with end support and with end mass. *Journal of the Acoustical Society of America* **55,** 986–991.

Mabie, H. H. and Rogers, C. B. (1972). Transverse vibrations of double-tapered cantilever beams. *Journal of the Acoustical Society of America* **51,** 1771–1774.

Mabie, H. H. and Rogers, C. B. (1968). Transverse vibrations of tapered cantilever beams with end support. *Journal of the Acoustical Society of America* **44,** 1739–1741.

Naguleswaran, S. (1994). A direct solution for the transverse vibration of Euler–Bernoulli wedge and cone beams. *Journal of Sound Vibration* **172,** 289–304.

Sanger, D. J. (1968). Transverse vibration of a class of non-uniform beams. *Journal of Mechanical Engineering Science* **10,** 111–120.

Sato, K. (1980). Transverse vibrations of linearly tapered beams with ends restrained elastically against rotation subjected to axial force. *International Journal of Mechanical Science* **22,** 109–115.

To, C. W. S. (1982). Vibration of a cantilever beam with a base excitation and tip mass. *Journal of Sound Vibration* **83,** 445–460.

Ward, P. F. (1913). Transverse vibration of bars of varying cross-section. *Philosophical Magazine* **25**(6), 85–106.

Watson, G. B. (1966). *A Treatise on the Theory of Bessel Functions*. Cambridge University Press.

Wolfram, S. (1993). Research Inc., *Mathematica*, Version 2.2. Wolfram Research, Champaign, IL.

Yang, K. Y. (1990). The natural frequencies of a non-uniform beam with tip mass and with translational and rotational springs. *Journal of Sound Vibration* **137,** 339–341.